

Coherence, Homotopy and 2-Theories

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Abstract

2-Theories are a canonical way of describing categories with extra structure. 2-theory-morphisms are used when discussing how one structure can be replaced with another structure. This is central to categorical coherence theory. We place a Quillen model category structure on the category of 2-theories and 2-theory-morphisms where the weak equivalences are biequivalences of 2-theories. A biequivalence of 2-theories (Morita equivalence) induces and is induced by a biequivalence of 2-categories of algebras. This model category structure allows one to talk of the homotopy of 2-theories and discuss the universal properties of coherence.

1 Introduction

The history of coherence theory has its roots in homotopy theory. Saunders Mac Lane's foundational paper [19] on coherence theory was an abstraction of earlier work by James Stasheff on H-spaces [24] and by D.B.A. Epstein on Steenrod operations. Coherence theory went on to become an important part of many diverse areas of computer science and mathematics. Questions of categorical coherence arise in, to name but a few areas, linear logic, proof theory, concurrency theory, low-dimensional topology, quantum groups and quantum field theory. Although coherence theory has become a mature and independent part of category theory, it has always had a distinct homotopy theory flavor. One has the feeling that a monoidal category is the same “up to homotopy” as a strict monoidal category. Or that a braided tensor category can be “deformed” to a strict braided tensor category. This paper is a step toward clarifying and formulating the exact relationship between coherence theory and homotopy theory.

We shall use the language of algebraic 2-theories to talk about coherence. Algebraic 2-theories are a generalization of Lawvere's algebraic (1-)theories which

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are central to his functorial semantics [17]. Algebraic 1-theories are a categorical description of sets (or topological spaces, manifolds, vector spaces, etc) with extra equational structure. Algebraic 2-theories are 2-categorical descriptions of categories (or any objects in a 2-category) with extra structure. Whereas the 1-cells in a 1-theory correspond to operations on sets, the 1-cells in a 2-theory correspond to functors. 2-cells in a 2-theory correspond to natural transformations between functors. So we have the 2-theory of monoidal categories, \mathbf{T}_{Mon} , strict monoidal categories \mathbf{T}_{sMon} , braided categories \mathbf{T}_{Braid} , balanced categories \mathbf{T}_{Bal} etc (see [15] for many examples and uses.) Each 2-theory \mathbf{T} has an associated 2-category $\mathbf{2Alg}(\mathbf{T}, \mathbf{Cat})$ of algebras, morphisms between algebras and natural transformations between morphisms.

John Gray was the first to define 2-theories [9]. They are also discussed in Bloom *et al* [4] and John Powers [20, 21] (although our definition of morphisms between algebras is slightly different from [21])

Coherence theory is concerned with the way one algebraic structure interacts with another. Let $F : \mathbf{T}_1 \rightarrow \mathbf{T}_2$ be a 2-theory-morphism. F induces a $F^* : \mathbf{2Alg}(\mathbf{T}_2, \mathbf{Cat}) \rightarrow \mathbf{2Alg}(\mathbf{T}_1, \mathbf{Cat})$ and a quasi-left adjoint $F_* : \mathbf{2Alg}(\mathbf{T}_1, \mathbf{Cat}) \rightarrow \mathbf{2Alg}(\mathbf{T}_2, \mathbf{Cat})$. The strength of this adjunction determines, and is determined by F . The adjunction can be an isomorphism, equivalence, biequivalence, strict quasi-adjunction etc. We shall be particularly interested in the notion of a biequivalence.

A biequivalence is a 2-categorical generalization of an equivalence. Whereas in an equivalence, the unit and counit are isomorphisms, in a biequivalence, the unit and counit are themselves equivalences. Examples of biequivalences are abundant. The phrase “Every monoidal category is tensor equivalent to a strict monoidal category” is a way of saying that $\mathbf{2Alg}(\mathbf{T}_{Mon}, \mathbf{Cat})$ and $\mathbf{2Alg}(\mathbf{T}_{sMon}, \mathbf{Cat})$ are biequivalent. We might call \mathbf{T}_{Mon} and \mathbf{T}_{sMon} Morita equivalent 2-theories. This fact is equivalent to saying that \mathbf{T}_{Mon} is biequivalent to \mathbf{T}_{sMon} . When we say that two types of categorical structure are the same “up to homotopy” or “are of the same homotopy type”, we mean there is a biequivalence between their 2-theories.

In 1967, Daniel Quillen [22] showed one how to talk about homotopy theory from a categorical point of view (see also [6, 13]). A category \mathbf{C} has a functorial closed Quillen model category (FCQMC) structure if there are three classes of morphisms in \mathbf{C} called weak equivalences, fibrations and cofibrations. These classes of maps must satisfy certain axioms that are of importance in homotopy theory. Examples of FCQMCs are the category of topological spaces, simplicial sets, simplicial complexes and \mathbf{Cat} . However many other categories like algebraic objects, chain complexes of algebraic objects, or spectra also have FCQMC structures. Once one has such a structure, one can go on to create the analogies of path spaces, mapping cylinders, Puppe sequences and all other important tools of homotopy theory. Given a FCQMC structure, one formally inverts the weak equivalences (that is, makes the weak equivalences into isomorphisms) in order to construct $Ho(\mathbf{C})$, the homotopy category of \mathbf{C} , and a

functor $\gamma : \mathbf{C} \longrightarrow Ho(\mathbf{C})$. γ has the universal property that given any category \mathbf{D} and functor $F : \mathbf{C} \longrightarrow \mathbf{D}$ that inverts the weak equivalences of \mathbf{C} , there is a unique $G : Ho(\mathbf{C}) \longrightarrow \mathbf{D}$ such that $G \circ \gamma = F$.

The goal of this paper is to show that the category of 2-theories and 2-theory-morphisms has a FCQMC structure. The weak equivalences will be biequivalences. It is important to realize that this paper is not a generalization of any known theorem about 1-theories. We know of no nontrivial FCQMC structure on the category of 1-theories (a FCQMC structure is trivial if the weak equivalences are exactly isomorphisms and hence $\mathbf{C} = Ho(\mathbf{C})$.) Only 2-theories have the flexibility to have a homotopy theory. 1-theories are too rigid for this. If one tries to do the same trick with 1-theories by making weak equivalences into genuine equivalences of 1-theories, one gets the trivial FCQMC structure. This follows from the fact that F is a genuine equivalence of 1-theories iff F is an isomorphism of 1-theories.

With the FCQMC in place, we can go on and write down universal properties of coherence.

This paper is organized as follows. Section 2 is a review of the relevant aspects of 2-theories. Section 3 is a discussion of the calculus of biequivalences. After defining the weak equivalences, fibrations, and cofibrations, Section 4 goes on to prove that they satisfy the axioms of an FCQMC structure. Some universal properties of coherence are discussed in Section 5. Section 6 is a look towards future directions that this project can take.

Notation. 2-Theories shall be denoted as $\mathbf{T}, \mathbf{T}', \mathbf{T}_1, \mathbf{T}_2, \dots$ 2-theory-morphisms are capital letters F, G, H, \dots . Lower-case Greek letters $\alpha, \beta, \gamma, \delta, \dots$ will denote 2-theory-natural transformations. Capital Greek letters Φ, Ψ, Ξ, \dots will denote 2-theory-modifications. The three compositions of morphisms in **2Theories** shall be denoted \circ_0, \circ_1 , and \circ_2 but will be omitted when no ambiguity arises. We shall write $x \in_i \mathbf{X}$ if x is an i -cell of the n -category \mathbf{X} where $i = 0, 1, 2, \dots, n$. 1-Categories will be shown in non-bold typeface while 2-categories are in bold typeface. Following Gray [8], we shall denote all 3-categories by placing a tilde above it.

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2 2-Theories

Let Fin_{sk} denote the skeletal category of finite sets. The 2-category $\overline{\mathbf{Fin}}_{sk}$ is Fin_{sk} with only identity 2-cells. Place a coproduct structure on $\overline{\mathbf{Fin}}_{sk}$. A coproduct structure for a 2-category is similar to a coproduct structure for a 1-category. However, there is an added requirement that for every finite family of 1-cells with common source and target, there is a 1-cell with injection 2-

cells that satisfy the obvious universal property. When we talk of preserving coproduct structures, we mean preserving the coproduct strictly (equality).

Definition 1 A (single sorted algebraic) **2-theory** is a 2-category \mathbf{T} with a given coproduct structure and a 2-functor $G_{\mathbf{T}} : \overline{\mathbf{Fin}}_{sk} \longrightarrow \mathbf{T}$ such that $G_{\mathbf{T}}$ is bijective on 0-cells and preserves the coproduct structure.

The following examples are well known.

Example.2.1: $\overline{\mathbf{Fin}}_{sk}$ is the initial 2-theory. Just as \mathbf{Fin}_{sk} is the theory of sets, so too, $\overline{\mathbf{Fin}}_{sk}$ is the theory of categories. \square

Example.2.2: Let \mathbf{T}_{Bin} be $\overline{\mathbf{Fin}}_{sk}$ with a nontrivial generating 1-cell $\otimes : 1 \longrightarrow 2$ thought of as a binary operation (bifunctor). \square

Example.2.3: \mathbf{T}_{Mon} is the 2-theory of monoidal (tensor) categories. It is a 2-theory “over” \mathbf{T}_{Bin} with a 1-cell $e : 1 \longrightarrow 0$. The isomorphic 2-cells are generated by

$$\begin{array}{ccc}
 & & 0 \amalg 1 \\
 & \nearrow \sim & \uparrow e \amalg id \\
 1 & \xrightarrow{\quad} & 1 \amalg 1 \\
 & \searrow \sim & \downarrow id \amalg e \\
 & & 1 \amalg 0
 \end{array}$$

(Note: The diagram above is a simplified representation of the one in the image. The image shows a more complex diagram with 2-cells λ and ρ and a 2-cell α between the two paths from 1 to $1 \amalg 1$.)

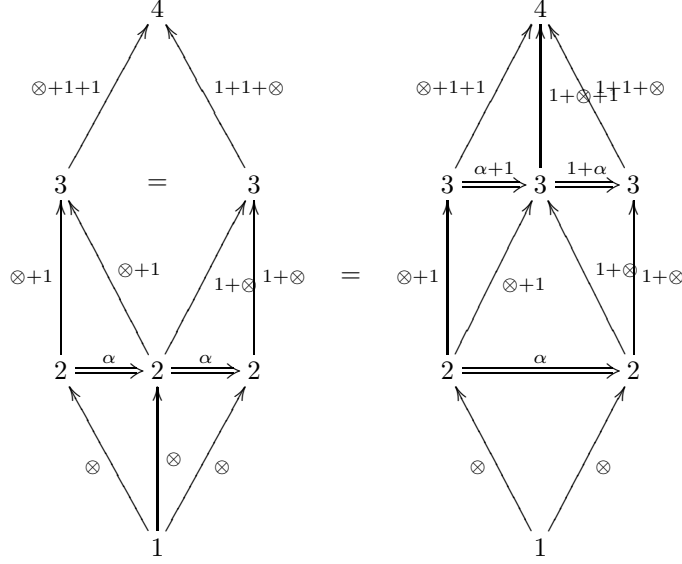
and

$$\begin{array}{ccccc}
 & & 1 & \xrightarrow{\otimes} & 1 + 1 \\
 & \nearrow \sim & & & \searrow \sim \\
 1 & & & & 1 + 1 \\
 \downarrow \otimes & & & & \downarrow 1 + \otimes \\
 1 + 1 & & & & 1 + 2 \\
 & \searrow \sim & & & \nearrow \sim \\
 & & 1 + 1 & \xrightarrow{\otimes + 1} & 2 + 1
 \end{array}$$

(Note: The diagram above is a simplified representation of the one in the image. The image shows a more complex diagram with 2-cells α and β and a 2-cell γ between the two paths from $1 + 1$ to $2 + 1$.)

where the corner isomorphisms $n + m \longrightarrow m + n$ is in $\overline{\mathbf{Fin}}_{sk}$. These 2-cells are subject to a unital equation (left for the reader) and the now-famous pentagon

condition:



(We leave out the corner isomorphisms in order to make the diagram easier to read. However they are important and must be placed in the definition). \square

Example.2.4: The theory of braided tensor categories \mathbf{T}_{Braid} and balanced tensor categories, \mathbf{T}_{Bal} , are easily described in a similar manner [15]. \square

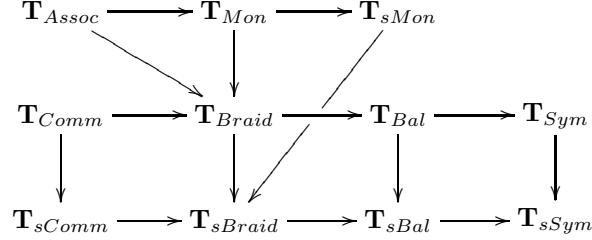
Example.2.5: Associative categories [27] which are monoidal categories in which the pentagon coherence does not necessarily hold are described by \mathbf{T}_{Assoc} . Similarly, commutative categories [28] which are braided tensor categories that do not necessarily satisfy the hexagon coherence condition are described by \mathbf{T}_{Comm} . \square

Example.2.6: Whenever we have a theory with strict associativity, we denote it with a small “s” followed by the usual name e.g. \mathbf{T}_{sMon} , \mathbf{T}_{sBraid} , \mathbf{T}_{sBal} etc. \square

Definition 2 A **2-theory-morphism** from \mathbf{T}_1 to \mathbf{T}_2 is a 2-functor $G : \mathbf{T}_1 \longrightarrow \mathbf{T}_2$ such that $G \circ G_{\mathbf{T}_1} = G_{\mathbf{T}_2}$. A **2-theory-natural transformation** is a natural transformation $\gamma : G_1 \Longrightarrow G_2$ between two 2-theory-morphisms. A **2-theory-modification** is a modification between two 2-theory-natural transformations.

We shall denote the 3-category of 2-theories, 2-theory-morphisms, 2-theory-natural transformations and 2-theory-modifications as **2Theories**.

Here is a diagram of some of the 2-theories and 2-theory morphisms that we will work with.



Many examples of 2-theories and their morphisms come from one dimensional theories in the following way. Let **Theories** denote the usual [17] 2-category of theories, theory-morphisms and theory-natural transformations. One can think of **Theories** as a 3-category $\widetilde{\mathbf{Theories}}$ with only trivial 3-cells. Analogous to the relationship between sets and topological spaces, we have the following adjunctions:

$$\begin{array}{ccc}
 & \xleftarrow{\pi_0} & \\
 & \perp d & \\
 \widetilde{\mathbf{Theories}} & \xrightarrow{\quad} & \widetilde{\mathbf{2Theories}} \\
 & \perp U & \\
 & \xleftarrow{c} &
 \end{array} \quad (1)$$

$c(T)$ is the 2-theory with the same 1-cells as T and a unique 2-cell between nontrivial 1-cells. $d(T)$ has the same 1-cells as T and only trivial 2-cells. $U(\mathbf{T})$ forgets the 2-cells of \mathbf{T} . $\pi_0(\mathbf{T})$ is a quotient theory of \mathbf{T} where two 1-cells are set equal if there is a 2-cell between them. These functors extend in an obvious way to 3-functors. By adjunction we mean a strict 3-adjunction; that is the universal property is satisfied by a strict 2-category isomorphism. For example the following 2-categories are isomorphic

$$Hom_{\widetilde{\mathbf{Theories}}}(T, U(\mathbf{T})) \cong Hom_{\widetilde{\mathbf{2Theories}}}(d(T), \mathbf{T})$$

Example.2.7: $\overline{\mathbf{Fin}}_{sk} = d(Fin_{sk})$, that is, the theory of categories is the discrete theory of sets. \square

Example.2.8: $\mathbf{T}_{Bin} = d(T_{Magmas})$. \square

Example.2.9: $d(T_{Monoids})$ is the theory of strict monoidal categories, \mathbf{T}_{sMon} . \square

Example.2.10: Let $T_{Magmas\bullet}$ be the theory of pointed magmas i.e. the theory of magmas with a distinguished element. $c(T_{Magmas\bullet})$ is the 2-theory of symmetric (monoidal) tensor categories. Warning: not all operations are supposed to be isomorphic to one another. In particular, the projections (inclusions) live in $\overline{\mathbf{Fin}}_{sk}$ and are not isomorphic. \square

Example.2.11: Let \mathbf{T}_{Braid} denote the 2-theory of braided tensor categories. $\pi_0(\mathbf{T}_{Braid})$ is the theory of commutative monoids. \square

The units and counits of these adjunctions are of interest. $\varepsilon : \pi_0 dT \rightarrow T$, $\mu : T \rightarrow UdT$ and $\varepsilon : UcT \rightarrow T$ are all identity theory-morphisms. More importantly, $\mu : \mathbf{T} \rightarrow d\pi_0 \mathbf{T}$ is the 2-theory-morphism corresponding to “strictification”. Every 2-cell becomes the identity. “Strictification” is often used in coherence theory. Similarly, $\mu : \mathbf{T} \rightarrow cU\mathbf{T}$ might be called “coherification” where a 2-theory is forced to be coherent. $\varepsilon : dU\mathbf{T} \rightarrow \mathbf{T}$ is the injection of the 1-theory into the 2-theory.

Definition 3 Given a 2-theory \mathbf{T} and a 2-category \mathbf{C} with a product structure, an **algebra** of \mathbf{T} in \mathbf{C} is a product preserving 2-functor $F : \mathbf{T}^{op} \rightarrow \mathbf{C}$.

Definition 4 A **quasi-natural transformation** [5, 8] σ from an algebra F to an algebra F' is

- A family of 1-cells in \mathbf{C} , $\sigma_n : F(n) \rightarrow F'(n)$ indexed by 0-cells of \mathbf{T} . This family must preserve products i.e. $\sigma_n = (\sigma_1)^n : F(1)^n \rightarrow F'(1)^n$.
- A family of 2-cells in \mathbf{C} , σ_f , indexed by 1-cells $f : m \rightarrow n$ of \mathbf{T} . σ_f makes the following diagram commute.

$$\begin{array}{ccccc}
 & & F(1)^n & \xrightarrow{\sigma^n} & F'(1)^n \\
 & \nearrow \sim & & & \searrow \sim \\
 F(n) & & & & F'(n) \\
 \downarrow Ff & & & & \downarrow F'f \\
 F(m) & & & & F'(m) \\
 & \searrow \sim & \nearrow \sigma_f & \nearrow \sigma_m & \nearrow \sim \\
 & & F(1)^m & \xrightarrow{\sigma^m} & F'(1)^m
 \end{array}$$

These morphisms must satisfy the following conditions:

1. If f is in the image of $G_{\mathbf{T}} : \overline{\mathbf{Fin}}_{sk} \rightarrow \mathbf{T}$, then $\sigma_f = id$. That is, the quasi-commutative diagram must commute strictly. This condition includes $\sigma_{id_n} = id_{\sigma_n}$.
2. σ preserves the (co)product structure: $\sigma_{f+f'} = \sigma_f \times \sigma_{f'}$. See [29] for an exact diagram.

3. $\sigma_{g \circ f} = \sigma_f \circ_v \sigma_g$ where $\circ_v = \circ_1$ is the vertical composition of 2-cells.
4. σ behaves well with respect to 2-cells of \mathbf{T} . See [29] for an exact diagram.

We shall call σ an **iso**-quasi-natural transformation if σ_f is an iso-2-cell for all $f \in {}_1 \mathbf{T}$.

Definition 5 Given two quasi-natural transformations $\sigma, \sigma' : F \longrightarrow F'$, a **modification** $\Sigma : \sigma \rightsquigarrow \sigma'$ from σ to σ' is a family of 2-cells $\Sigma_n : \sigma_n \Longrightarrow \sigma'_n$ indexed by the 0-cells of \mathbf{T} . These 2-cells must satisfy the following conditions:

1. Σ preserves products i.e. $\Sigma_n = (\Sigma_1)^n : (\sigma_1)^n \Longrightarrow (\sigma'_1)^n$.
2. Σ behaves well with respect to the 2-cells of \mathbf{T} . That is, if we have

$$m \begin{array}{c} \xrightarrow{f} \\ \Downarrow \alpha \\ \xrightarrow{f'} \end{array} n$$

then we have the following “cube relation”:

$$\begin{array}{ccc} F(n) & \xrightarrow{\sigma'_n} & F'(n) \\ \downarrow id & & \downarrow id \\ F(n) & \xrightarrow{\sigma_n} & F'(n) \\ & \nearrow \Sigma_n & \\ & F'(n) & \xrightarrow{F'(\alpha)} id \\ & \uparrow \sigma_f & \downarrow F'(f) \\ F(m) & \xrightarrow{\sigma_m} & F'(m) \end{array} \quad \begin{array}{ccc} F(n) & \xrightarrow{\sigma'_n} & F'(n) \\ & \searrow F(f') & \downarrow F'(f') \\ & F(m) & \xrightarrow{\sigma'_m} F'(m) \\ \downarrow id & & \downarrow id \\ F(n) & \xrightarrow{F(\alpha)} & id \\ & \nearrow \Sigma_m & \\ & F(m) & \xrightarrow{\sigma_m} F'(m) \end{array}$$

For a given 2-theory \mathbf{T} and a 2-category \mathbf{C} with a product structure, we denote the 2-category of algebras, quasi-natural transformations and modifications as $\mathbf{2Alg}(\mathbf{T}, \mathbf{C})$. We shall denote the locally full sub-2-category of algebras, **iso**-quasi-natural transformations and modifications as $\mathbf{2Alg}^i(\mathbf{T}, \mathbf{C})$.

(A quasi-natural transformation is a way of having an operations preserved up to a 2-cell. Many times in coherence theory, one wants some operations to be preserved up to a 2-cell and some operations to be preserved strictly. This is done in [29] with the notion of a relative quasi-natural transformation. We demand two 2-theories $\mathbf{T}_1, \mathbf{T}_2$ and a 2-theory-morphism between them $G : \mathbf{T}_1 \longrightarrow \mathbf{T}_2$. \mathbf{T}_1 controls which operations in \mathbf{T}_2 should be preserved strictly. A relative quasi natural transformation between two \mathbf{T}_2 algebras is a quasi-natural

transformation $\sigma : F \Rightarrow F'$ such that $\sigma \circ G^{op}$ is a natural transformation (not quasi) from $F \circ G^{op}$ to $F' \circ G^{op}$.

$$\mathbf{T}_1^{op} \xrightarrow{G^{op}} \mathbf{T}_2^{op} \begin{array}{c} \xrightarrow{F} \\ \Downarrow \sigma \\ \xrightarrow{F'} \end{array} \mathbf{C}.$$

$\mathbf{2Alg}_{\mathbf{G}}(\mathbf{T}_2, \mathbf{C})$ has the same algebras as $\mathbf{2Alg}(\mathbf{T}_2, \mathbf{C})$ but with only relative quasi-natural transformations between them. The present paper will not use this notion. However, we will discuss the relationship between relative quasi-natural transformations and relative homotopy theory in Section 6.)

Fixing $\mathbf{C} = \mathbf{Cat}$ we can extend $\mathbf{2Alg}(-, \mathbf{Cat})$ to be a 3-functor from $\mathbf{2Theories}^{op}$ to $\mathbf{2Cat}/\mathbf{Cat}$ in the obvious way.

It is common to look at the algebras of one theory in the category of algebras of another theory. The theory of such algebras is given as the Kronecker product of the two theories.

The Kronecker product of (1-)theories is a well understood coherent symmetric monoidal 2-bifunctor $\otimes_K : \mathbf{Theories} \times \mathbf{Theories} \longrightarrow \mathbf{Theories}$. Let T_1 and T_2 be two theories. $T_1 \otimes_K T_2$ is a theory that satisfies the universal property

$$Alg(T_1 \otimes_K T_2, C) \cong Alg(T_1, Alg(T_2, C)).$$

$T_1 \otimes_K T_2$ is constructed as follows. Construct the coproduct $T_1 \coprod T_2$ in the category of theories (pushout in \mathbf{Cat} .) Place a congruence on $T_1 \coprod T_2$ such that for all $f : m \longrightarrow m'$ in T_1 and $g : n \longrightarrow n'$ in T_2 the diagram

$$\begin{array}{ccccc} & & n^m & \xrightarrow{g^m} & n'^m \\ & \nearrow \sim & & & \searrow \sim \\ m^n & & & & m^{n'} \\ \downarrow f^n & & & & \downarrow f^{n'} \\ m'^n & & & & m'^{n'} \\ & \searrow \sim & n^{m'} & \xrightarrow{g^{m'}} & n'^{m'} \\ & & & & \nearrow \sim \end{array}$$

commutes. We have a full theory-morphism $T_1 \coprod T_2 \longrightarrow T_1 \otimes_K T_2$.

We will work with a two-dimensional analogue to the Kronecker product. (See [29] for more details.)

Definition 6 A **(2-)Kronecker product** of 2-theories is a 3-bifunctor

$$\otimes^K : \widetilde{\mathbf{2Theories}} \times \widetilde{\mathbf{2Theories}} \longrightarrow \widetilde{\mathbf{2Theories}}$$

that satisfies the following universal property: for all

$$\mathbf{T}_1 \xleftarrow{G_{\mathbf{T}_1}} \overline{\mathbf{Fin}}_{sk} \xrightarrow{G_{\mathbf{T}_2}} \mathbf{T}_2$$

there is an induced

$$\begin{array}{ccc} \overline{\mathbf{Fin}}_{sk} & \xrightarrow{G_{\mathbf{T}_1}} & \mathbf{T}_1 \\ \downarrow G_{\mathbf{T}_2} & \searrow G_{\mathbf{T}_1} \otimes^K G_{\mathbf{T}_2} & \downarrow \\ \mathbf{T}_2 & \xrightarrow{\quad} & \mathbf{T}_1 \otimes^K \mathbf{T}_2 \end{array}$$

and for all 2-categories with finite products \mathbf{C} , an isomorphism of 2-categories

$$\mathbf{2Alg}(\mathbf{T}_1 \otimes^K \mathbf{T}_2, \mathbf{C}) \cong \mathbf{2Alg}(\mathbf{T}_1, \mathbf{2Alg}(\mathbf{T}_2, \mathbf{C}))$$

which is natural for all cells in $\widetilde{\mathbf{2Theories}}$ and \mathbf{C} .

It will be helpful to examine the naturality conditions in terms of 1-cells of $\widetilde{\mathbf{2Theories}}$. Let $F_1 : \mathbf{T}_1 \longrightarrow \mathbf{T}'_1$ and $F_2 : \mathbf{T}_2 \longrightarrow \mathbf{T}'_2$ be 2-theory morphisms. By the functoriality of \otimes^K there is an induced 2-theory-morphism $F_1 \otimes^K F_2 : \mathbf{T}_1 \otimes^K \mathbf{T}_2 \longrightarrow \mathbf{T}'_1 \otimes^K \mathbf{T}'_2$ such that

$$\begin{array}{ccc} \mathbf{2Alg}(\mathbf{T}'_1 \otimes^K \mathbf{T}'_2, \mathbf{C}) & \xrightarrow{\cong} & \mathbf{2Alg}(\mathbf{T}'_1, \mathbf{2Alg}(\mathbf{T}'_2, \mathbf{C})) \\ \downarrow \mathbf{2Alg}(F_1 \otimes^K F_2, \mathbf{C}) & & \downarrow \mathbf{2Alg}(F_1, \mathbf{2Alg}(F_2, \mathbf{C})) \\ \mathbf{2Alg}(\mathbf{T}_1 \otimes^K \mathbf{T}_2, \mathbf{C}) & \xrightarrow{\cong} & \mathbf{2Alg}(\mathbf{T}_1, \mathbf{2Alg}(\mathbf{T}_2, \mathbf{C})) \end{array}$$

In order to construct $\mathbf{T}_1 \otimes^K \mathbf{T}_2$, we take the coproduct $\mathbf{T}_1 \coprod_{\overline{\mathbf{Fin}}_{sk}} \mathbf{T}_2$ in $\widetilde{\mathbf{2Theories}}$ and we freely add in the following 2-cells: For every $f : m' \longrightarrow m$ in \mathbf{T}_1 and $g : n' \longrightarrow n$ in \mathbf{T}_2 we add the 2-cell $\delta_{\mathbf{T}_1, \mathbf{T}_2}(f, g)$ that makes the

following diagram commute:

$$\begin{array}{ccccc}
 & & n^m & \xrightarrow{g^m} & n'^m \\
 & \nearrow \sim & & & \searrow \sim \\
 m^n & & & & m'^{n'} \\
 \downarrow f^n & & \delta_{\mathbf{T}_1, \mathbf{T}_2}(f, g) & & \downarrow f^{n'} \\
 m'^n & & & & m'^{n'} \\
 & \searrow \sim & n'^{m'} & \xleftarrow{g^{m'}} & n'^{m'} \\
 & & & & \nearrow \sim
 \end{array}
 \quad (2)$$

The δ 's must satisfy the following coherence conditions that are compatible to the four coherence conditions in the definition of a quasi-natural transformation.

1. If f is in the image of G_1 , then $\delta(f, g)$ must be set to the identity.
2. δ must preserve products in f .
3. $\delta(f \circ f', g) = \delta(f, g) \circ_v \delta(f', g)$
4. δ must preserve 2-cells.

The fact that there is choice in the construction of $\mathbf{T}_1 \otimes^K \mathbf{T}_2$, should not disturb the reader since we never claimed that $\mathbf{T}_1 \otimes^K \mathbf{T}_2$ should be unique. Rather, it should be unique up to a (2-)isomorphism. In order to see that our construction of $\mathbf{T}_1 \otimes^K \mathbf{T}_2$ satisfies the universal properties demanded of it, we must realize that our construction was made to mimic the definition of a quasi-natural transformation in our 2-categories of algebras.

Given $F_1 : \mathbf{T}_1 \longrightarrow \mathbf{T}'_1$ and $F_2 : \mathbf{T}_2 \longrightarrow \mathbf{T}'_2$, we construct $F_1 \otimes^K F_2$ as follows: construct $F_1 \amalg F_2 : \mathbf{T}_1 \amalg \mathbf{T}_2 \longrightarrow \mathbf{T}'_1 \amalg \mathbf{T}'_2$ and define $F_1 \otimes^K F_2$ on $\delta_{\mathbf{T}_1, \mathbf{T}_2}(f, g)$ as

$$(F_1 \otimes^K F_2)(\delta_{\mathbf{T}_1, \mathbf{T}_2}(f, g)) = \delta_{\mathbf{T}'_1, \mathbf{T}'_2}(F_1(f), F_2(g)). \quad (3)$$

3 Biequivalences

The notion of a biequivalence is central to this paper. A biequivalence is a 2-categorical generalization of an equivalence. We know of no original sources

for the idea or the name. The properties of biequivalences seem to be well known folklore that remains unwritten. We shall be explicit with some of these properties.

Definition 7 *Within a 2-category \mathbf{A} , a 1-cell $f : a \longrightarrow a'$ is an **equivalence** if there exists a 1-cell $g : a' \longrightarrow a$ and isomorphic 2-cells $\eta : id_a \xrightarrow{\sim} g \circ f$ and $\varepsilon : f \circ g \xrightarrow{\sim} id_{a'}$. We denote such an equivalence as $(f, g, \eta, \varepsilon) : a \longrightarrow a'$.*

Definition 8 *Within a 3-category $\tilde{\mathbf{C}}$, a 1-cell $F : \mathbf{A} \longrightarrow \mathbf{B}$ is a **biequivalence** if there is a 1-cell $G : \mathbf{B} \longrightarrow \mathbf{A}$ with an equivalence $(\eta, \delta, \Phi, \Xi) : id_{\mathbf{A}} \longrightarrow G \circ F$ in $\tilde{\mathbf{C}}(\mathbf{A}, \mathbf{A})$ and an equivalence $(\varepsilon, \zeta, \Psi, \Omega) : F \circ G \longrightarrow id_{\mathbf{B}}$ in $\tilde{\mathbf{C}}(\mathbf{B}, \mathbf{B})$.*

Let us look at this definition in more detail. $F : \mathbf{A} \longrightarrow \mathbf{B}$ is a biequivalence if F is part of a 10-tuple

$$(F, G, \eta, \delta, \varepsilon, \zeta, \Phi, \Xi, \Psi, \Omega) : \mathbf{A} \longrightarrow \mathbf{B}$$

where

$$F : \mathbf{A} \longrightarrow \mathbf{B} \qquad G : \mathbf{B} \longrightarrow \mathbf{A}$$

$$\eta : id_{\mathbf{A}} \longrightarrow G \circ F \qquad \delta : G \circ F \longrightarrow id_{\mathbf{A}} \qquad \varepsilon : F \circ G \longrightarrow id_{\mathbf{B}} \qquad \zeta : id_{\mathbf{B}} \longrightarrow F \circ G$$

$$\Phi : id_{\mathbf{A}} \xrightarrow{\sim} \delta \circ \eta \quad \Xi : \eta \circ \delta \xrightarrow{\sim} id_{G \circ F} \qquad \Psi : id_{F \circ G} \xrightarrow{\sim} \zeta \circ \varepsilon \quad \Omega : \varepsilon \circ \zeta \xrightarrow{\sim} id_{\mathbf{B}}$$

(Ross Street [25] has written a general definition of a k -equivalence between two n -categories where $k \leq n + 1$. Using that language, a biequivalence is a 3-equivalence of two 2-categories. See [30] for general properties of a k -equivalence.)

Proposition 1 *Let $(F, G, \eta, \delta, \varepsilon, \zeta, \Phi, \Xi, \Psi, \Omega) : \mathbf{A} \longrightarrow \mathbf{B}$ be a biequivalence. The following are also biequivalences.*

- 1) $(F, G, \delta, \eta, \varepsilon, \zeta, \Xi^{-1}, \Phi^{-1}, \Psi, \Omega) : \mathbf{A} \longrightarrow \mathbf{B}$
- 2) $(F, G, \eta, \delta, \zeta, \varepsilon, \Phi, \Xi, \Omega^{-1}, \Psi^{-1}) : \mathbf{A} \longrightarrow \mathbf{B}$
- 3) $(G, F, \zeta, \varepsilon, \delta, \eta, \Psi, \Omega, \Phi, \Xi) : \mathbf{B} \longrightarrow \mathbf{A}$

Proposition 2 *Let $(F, G, \eta, \delta, \varepsilon, \zeta, \Phi, \Xi, \Psi, \Omega) : \mathbf{A} \longrightarrow \mathbf{B}$ and $(F', G', \eta', \delta', \varepsilon', \zeta', \Phi', \Xi', \Psi', \Omega') : \mathbf{B} \longrightarrow \mathbf{C}$ be biequivalences. Then $(F' \circ F, G \circ G', (G\eta'F) \circ \eta, \delta \circ (G\delta'F), \varepsilon' \circ (F'\varepsilon G'), (F'\zeta G') \circ \zeta', \Phi'', \Xi'', \Psi'', \Omega'') : \mathbf{A} \longrightarrow \mathbf{C}$ is also a biequivalence where*

$$\begin{aligned} \Phi'' &= [(\delta G \delta') \zeta (\eta' F \eta)] \circ [(\delta G) \Phi' (F \eta)] \circ [(\delta) \eta (\eta)] \circ \Phi \\ \Xi'' &= \delta \circ [(G) \Xi' (F)] \circ [(G \eta') \varepsilon (\delta' F)] \circ [(G \eta' F) \Xi (G \delta' F)] \\ \Psi'' &= [(F' \zeta G') \Psi' (F' \varepsilon G')] \circ [(F' \zeta) \eta (\varepsilon G')] \circ [(F') \Psi (G')] \circ \varepsilon' \\ \Omega'' &= \Omega' \circ [(\varepsilon') \varepsilon (\zeta')] \circ [(\varepsilon' F) \Omega (G' \zeta')] \circ [(\varepsilon' F \varepsilon) \delta' (\zeta G' \zeta')]. \end{aligned}$$

Proposition 3 Let $(F_1, G_1, \eta_1, \delta_1, \varepsilon_1, \zeta_1, \Phi_1, \Xi_1, \Psi_1, \Omega_1) : \mathbf{A}_1 \longrightarrow \mathbf{B}_1$ and

$$(F_2, G_2, \eta_2, \delta_2, \varepsilon_2, \zeta_2, \Phi_2, \Xi_2, \Psi_2, \Omega_2) : \mathbf{A}_2 \longrightarrow \mathbf{B}_2$$

be biequivalences. Then

$$((F_1 \amalg F_2), (G_1 \amalg G_2), (\eta_1 \amalg \eta_2), (\delta_1 \amalg \delta_2), (\varepsilon_1 \amalg \varepsilon_2), (\zeta_1 \amalg \zeta_2),$$

$$(\Phi_1 \amalg \Phi_2), (\Xi_1 \amalg \Xi_2), (\Psi_1 \amalg \Psi_2), (\Omega_1 \amalg \Omega_2)) : (\mathbf{A}_1 \amalg \mathbf{A}_2) \longrightarrow (\mathbf{B}_1 \amalg \mathbf{B}_2)$$

is also a biequivalence. (Similarly for products.)

Proposition 4 Let $F : \mathbf{A} \longrightarrow \mathbf{B}$ be a biequivalence and $G : \mathbf{B} \longrightarrow \mathbf{A}$ be a 2-functor such that

1) $G \circ F = id_{\mathbf{A}}$ (i.e. F is the inclusion of a full sub-2-category) then there is a biequivalence $(F, G, id, id, \varepsilon, \zeta, id, id, \Psi, \Omega) : \mathbf{A} \longrightarrow \mathbf{B}$

2) $F \circ G = id_{\mathbf{B}}$ (i.e. F is a surjection of 2-categories) then there is a biequivalence $(F, G, \eta, \delta, id, id, \zeta, \Phi, \Xi, id, id) : \mathbf{A} \longrightarrow \mathbf{B}$

3) $G \circ F \cong id_{\mathbf{A}}$ (i.e. F is “almost” the inclusion of a full sub-2-category) then there is a biequivalence $(F, G, \eta, \eta^{-1}, \varepsilon, \zeta, id, id, \Psi, \Omega) : \mathbf{A} \longrightarrow \mathbf{B}$

4) $G \circ F \cong id_{\mathbf{B}}$ (i.e. F is “almost” a surjection of 2-categories) then there is a biequivalence $(F, G, \eta, \delta, \varepsilon, \varepsilon^{-1}, \Phi, \Xi, id, id) : \mathbf{A} \longrightarrow \mathbf{B}$

Definition 9 A biequivalence is called an **adjoint biequivalence** if

$$F\eta \cong \zeta F \quad F\delta \cong \varepsilon F \quad G\zeta \cong \eta G \quad G\varepsilon \cong \delta G$$

Proposition 5 If $(F, G, \eta, \delta, \varepsilon, \zeta, \Phi, \Xi, \Psi, \Omega) : \mathbf{A} \longrightarrow \mathbf{B}$ is a biequivalence then the following are adjoint biequivalences

1) $(F, G, \eta', \delta', \varepsilon, \zeta, \Phi', \Xi', \Psi, \Omega) : \mathbf{A} \longrightarrow \mathbf{B}$ where

$$\eta'_a = \delta_{GFa} \circ G\zeta_{Fa} \circ \eta_a$$

$$\delta'_a = \delta_a \circ G\varepsilon_{Fa} \circ \eta_{GFa}$$

$$\Phi'_a = [\delta_a \circ G\varepsilon_{Fa} \circ \Xi_{GFa}^{-1} \circ G\zeta_{Fa} \eta_a] \circ [\delta_a \circ G\Omega_{Fa}^{-1} \eta_a] \circ \Phi_a$$

$$\Xi'_a = \Phi_{GFa}^{-1} \circ [\delta_{GFa} \circ G\Psi_{Fa}^{-1} \circ \eta_{GFa}] \circ [\delta_{GFa} \circ G\zeta_{Fa} \circ \Xi_a \circ G\varepsilon_{Fa} \circ \eta_{GFa}]$$

2) $(F, G, \eta, \delta, \varepsilon', \zeta', \Phi, \Xi, \Psi', \Omega') : \mathbf{A} \longrightarrow \mathbf{B}$ where

$$\varepsilon'_b = \varepsilon_b \circ F\delta_{Gb} \circ \zeta_{FGb}$$

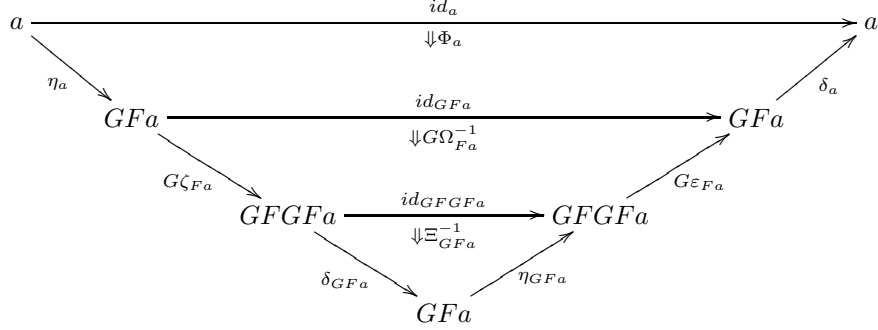
$$\zeta'_b = \varepsilon_{FGb} \circ F\eta_{Gb} \circ \zeta_b$$

$$\Phi'_b = [\varepsilon_{FGb} \circ F\eta_{Gb} \circ \Psi_b^{-1} \circ F\delta_{Gb} \zeta_{FGb}] \circ [\varepsilon_{FGb} \circ F\Xi_{Gb}^{-1} \zeta_{FGb}] \circ \Omega_{FGb}^{-1}$$

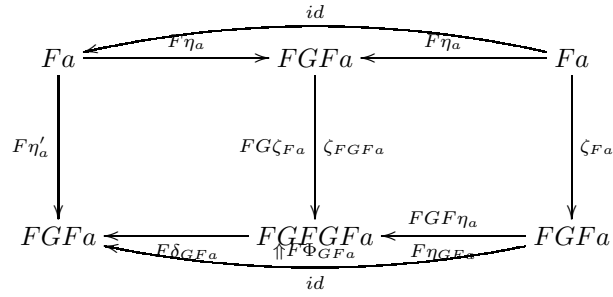
$$\Omega'_b = \Omega_b \circ [\varepsilon_b \circ F\Phi_{Gb}^{-1} \circ \zeta_b] \circ [\varepsilon_b \circ F\delta_{Gb} \circ \Psi_{FGb}^{-1} \circ F\eta_{Gb} \circ \zeta_b]$$

In other words, Any biequivalence can be made into an adjoint biequivalence.

Proof. We shall focus on part 1). Part 2) is completely symmetric. The following diagram will help the reader see Φ' more clearly.



Similar diagrams can be drawn for Ξ', Ψ' and Ω' . In order to see that this new biequivalence satisfies the first adjointness condition ($F\eta' \cong \zeta_F$), consider the following diagram



Where the left square commutes by the definition of $F\eta'_a$ and the right square commutes by the naturality of ζ . In detail:

$$\begin{aligned}
 F\eta'_a &= F\delta_{GFa} \circ FG\zeta_{Fa} \circ F\eta_a && \text{by definition of } \eta' \\
 &= F\delta_{GFa} \circ F\eta_{GFa} \circ \eta_{Fa} && \text{by naturality of } \zeta \\
 &\cong id \circ \zeta_{Fa} && \text{by } F\Phi_{GFa}^{-1} \\
 &= \zeta_{Fa}.
 \end{aligned}$$

The following three diagrams show the other three adjointness conditions:

$$\begin{array}{ccccc}
 & & id & & \\
 & \nearrow F\eta_{GFa} & \Downarrow F\Phi_{GFa} & \nwarrow F\delta_{GFa} & \\
 FGFa & \xrightarrow{\quad} & FGFGFa & \xrightarrow{\quad} & FGFa \\
 \downarrow F\delta'_a & & \downarrow FG\varepsilon_{Fa} \quad \varepsilon_{FGFa} & & \downarrow \varepsilon_{Fa} \\
 Fa & \xleftarrow{\quad F\delta_a \quad} & FGFa & \xrightarrow{\quad F\delta_a \quad} & Fa
 \end{array}$$

$$\begin{array}{ccccc}
 & & id & & \\
 & \nearrow \eta_{Gb} & \Downarrow \Phi_{Gb} & \nwarrow \delta_{Gb} & \\
 Gb & \xrightarrow{\quad} & GFGb & \xrightarrow{\quad} & Gb \\
 \downarrow \eta'_{Gb} & & \downarrow G\zeta_{FGb} \quad GFG\zeta_b & & \downarrow G\zeta_b \\
 GFGb & \xleftarrow{\quad GF\delta_{Gb} \quad} & GFGFGb & \xrightarrow{\quad \delta_{GFGb} \quad} & GFGb
 \end{array}$$

$$\begin{array}{ccccc}
 GFGB & \xrightarrow{\quad GF\eta_{Gb} \quad} & GFGB & \xleftarrow{\quad GF\eta_{Gb} \quad} & GFGB \\
 \downarrow \delta'_{Gb} & & \downarrow G\varepsilon_{FGb} \quad GFG\varepsilon_b & & \downarrow G\varepsilon_b \\
 Gb & \xleftarrow{\quad \delta_{Gb} \quad} & GFGb & \xleftarrow{\quad \eta_{Gb} \quad} & Gb \\
 & & id & &
 \end{array}$$

□

Proposition 6 *Let \mathbf{A} and \mathbf{B} be 2-categories and $F : \mathbf{A} \longrightarrow \mathbf{B}$ be a 2-functor. F is a biequivalence iff*

- 1) *for all $b \in_0 \mathbf{B}$, there exists an $a \in_0 \mathbf{A}$ such that $F(a)$ is equivalent to b and*
- 2) *for all $a, a' \in_0 \mathbf{A}$, $F : \mathbf{A}(a, a') \longrightarrow \mathbf{B}(F(a), F(a'))$ is an equivalence.*

Proof. Assume F is part of a an adjoint biequivalence then for all $b \in \mathbf{B}$, $Gb \in_0 \mathbf{A}$ and we have the equivalence $(\varepsilon_b, \zeta_b, \Psi_b, \Omega_b) : FG(b) \longrightarrow b$. The equivalence for part 2) is given as

$$(F, (\delta_{a'} \circ (-) \circ \eta_a) \circ G, \Phi_{a'}, \Omega_{Fa'}^{-1}) : \mathbf{A}(a, a') \longrightarrow \mathbf{B}(F(a), F(a')).$$

In order to understand the unit of this equivalence, consider the following diagram:

$$\begin{array}{ccccc}
 & & id & & \\
 & \nearrow \eta_a & \Downarrow \Phi_a & \searrow \delta_a & \\
 a & \xrightarrow{\quad} & GFa & \xrightarrow{\quad} & a \\
 \downarrow f & & \downarrow GFf & & \downarrow f \\
 a' & \xrightarrow{\quad} & GFa' & \xrightarrow{\quad} & a' \\
 & \nwarrow \eta_{a'} & \Uparrow \Phi_{a'} & \swarrow \delta_{a'} & \\
 & & id & &
 \end{array}$$

Let $f \in \mathbf{A}(a, a')$, then

$$\begin{aligned}
 f &= id_{a'} \circ f \\
 &\cong \delta_{a'} \circ \eta_{a'} \circ f \quad \text{by } \Phi_{a'} \\
 &= \delta_{a'} \circ GFf \circ \eta_a \quad \text{by naturality of } \eta \\
 &= [(\delta_{a'} \circ (-) \circ \eta_a) \circ G \circ F]f \quad \text{in } \mathbf{A}(a, a').
 \end{aligned}$$

In order to understand the counit of this equivalence, consider the following diagram:

$$\begin{array}{ccccc}
 & & id_F & & \\
 & \nearrow \zeta_{Fa} & \Uparrow \Omega_{Fa} & \searrow \varepsilon_{Fa} & \\
 Fa & \xrightarrow{\quad} & FGFa & \xrightarrow{\quad} & Fa \\
 \downarrow g & & \downarrow FGg & & \downarrow g \\
 Fa' & \xrightarrow{\quad} & FGFa' & \xrightarrow{\quad} & Fa' \\
 & \nwarrow \zeta_{Fa'} & \Downarrow \Omega_{Fa'} & \swarrow \varepsilon_{Fa'} & \\
 & & id & &
 \end{array}$$

Let $g \in \mathbf{B}(Fa, Fa')$, then

$$\begin{aligned}
 g &= id_{Fa'} \circ g \\
 &\cong \varepsilon_{Fa'} \circ \zeta_{Fa'} \circ g \quad \text{by } \Omega_{Fa'}^{-1} \\
 &= \varepsilon_{Fa'} \circ FGg \circ \zeta_{Fa} \quad \text{by naturality of } \zeta \\
 &\cong F\delta_{a'} \circ FGg \circ F\eta_a \quad \text{by adjoint biequivalence} \\
 &= F \circ (\delta_{a'} \circ G(-) \circ \eta_a)(g) \quad \text{in } \mathbf{B}(Fa, Fa').
 \end{aligned}$$

Conversely, assume that for all $b \in_0 \mathbf{B}$ there is an $a_b \in_0 \mathbf{A}$ such that $(f_b, G_b, \eta_b, \varepsilon_b) : F(a_b) \rightarrow b$ is an equivalence and for all $a, a' \in_0 \mathbf{A}$ there is an equivalence

$$(f_{a,a'}, g_{a,a'}, \eta_{a,a'}, \varepsilon_{a,a'}) : \mathbf{A}(a, a') \rightarrow \mathbf{B}(Fa, Fa').$$

Then we shall define $G : \mathbf{B} \longrightarrow \mathbf{A}$ as follows

$$\begin{aligned} G(b) &= a_b \\ G(h : b \longrightarrow b') &= g_{a_b, a_{b'}}(g_{b'} \circ h \circ f_b) \\ G(\beta : h \longrightarrow h') &= g_{a_b, a_{b'}}(g_{b'} \circ \beta \circ f_b) \end{aligned}$$

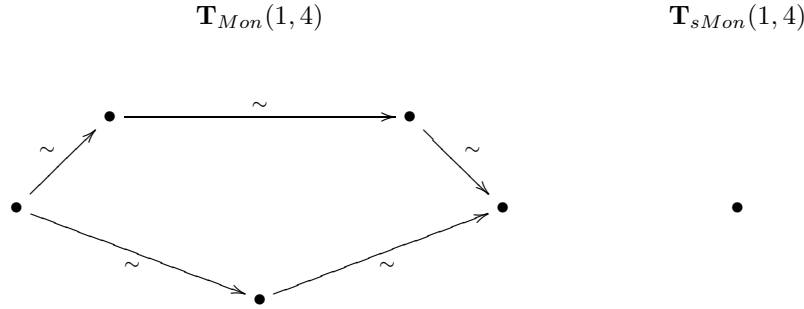
Define $\alpha : id_{\mathbf{A}} \longrightarrow G \circ F$ and $\gamma : F \circ G \longrightarrow id_{\mathbf{B}}$ as

$$\alpha_a = g_{a, GFa}(g_{Fa}) \quad \gamma_b = g_{a, GFa}(f_{Fa}).$$

$$\begin{aligned} (\gamma \circ \alpha)_a &= \gamma_a \circ \alpha_a \\ &= g_{a, GFa}(f_{Fa}) \circ g_{a, GFa}(g_{Fa}) \\ &= g_{a, GFa}(f_{Fa} \circ g_{Fa}) \\ &\cong g_{a, GFa}(id_{Fa}) \quad \text{by } \varepsilon_{Fa} \\ &= id. \end{aligned}$$

Similarly for $\alpha \circ \gamma$. Hence we have the sought-after biequivalence. \square

Let us restrict our attention from general 2-functors to 2-theory-morphisms. Since 2-theory-morphisms are bijective on 0-cells, by Proposition 6 we have that $F : \mathbf{T}_1 \longrightarrow \mathbf{T}_2$ is a biequivalence iff $F : \mathbf{T}_1(m, n) \longrightarrow \mathbf{T}_2(m, n)$ is an equivalence of categories for all $m, n \in \mathbf{N}$. A nice example of this is when $\mathbf{T}_1 = \mathbf{T}_{Mon}$, $\mathbf{T}_2 = \mathbf{T}_{sMon}$, $m = 1$ and $n = 4$. We then have the following two equivalent categories.



(Notice the importance of the morphisms being isomorphisms. This tells us that the Mac Lane's coherence theorem would not apply for categories with a tensor product and a non-isomorphism $A \otimes (B \otimes C) \longrightarrow (A \otimes B) \otimes C$ satisfying the pentagon condition.)

Similarly, there are obvious biequivalences between the 2-theory of braided monoidal categories and the 2-theory of strictly associative braided monoidal categories.

One can go on to prove a more general statement: For any algebraic 2-theory \mathbf{T}_X with an inclusion of \mathbf{T}_{Mon} , we have the following pushout

$$\begin{array}{ccc}
\mathbf{T}_{Mon} & \hookrightarrow & \mathbf{T}_X \\
\downarrow & & \downarrow \\
\mathbf{T}_{sMon} & \hookrightarrow & \mathbf{T}_{sX}
\end{array}$$

where \mathbf{T}_{sX} is the 2-theory \mathbf{T}_X with strict associativity. Since the left-hand side of the pushout is a biequivalence, the right-hand side is also.

The importance of biequivalences for the semantics of coherence theory is the following proposition.

Proposition 7 $F : \mathbf{T}_1 \longrightarrow \mathbf{T}_2$ is a biequivalence iff

$$F^* = \mathbf{2Alg}^i(F, \mathbf{Cat}) : \mathbf{2Alg}^i(\mathbf{T}_2, \mathbf{Cat}) \longrightarrow \mathbf{2Alg}^i(\mathbf{T}_1, \mathbf{Cat})$$

is a biequivalence.

Proof. If F is a biequivalence, then simply by the 3-functoriality of $F^* = \mathbf{2Alg}^i(-, \mathbf{Cat})$ the conclusion follows.

Conversely, assume F^* is a biequivalence. By Proposition 5, we may assume that F^* is an adjoint biequivalence. In [29] Proposition 2, we proved that for any 2-theory \mathbf{T} , $\mathbf{T}^{\mathbf{op}}(n, -) : \mathbf{T} \longrightarrow \mathbf{Cat}$ is the free \mathbf{T} -algebra on n generators. Since F^* is an adjoint biequivalence, it is not hard to show that

$$F^*(\mathbf{T}^{\mathbf{op}}_2(n, -)) \cong \mathbf{T}^{\mathbf{op}}_1(n, -).$$

In order to show that $F : \mathbf{T}_1 \longrightarrow \mathbf{T}_2$ is a biequivalence, it suffices to show that for all $m, n \in \mathbf{N}$, $F : \mathbf{T}_1(m, n) \longrightarrow \mathbf{T}_2(m, n)$ is an equivalence (Proposition 6).

$$\begin{aligned}
\mathbf{T}^{\mathbf{op}}_2(m, n) &\simeq \mathbf{2Alg}^i(\mathbf{T}_2, \mathbf{Cat})(\mathbf{T}^{\mathbf{op}}_2(m, -), \mathbf{T}^{\mathbf{op}}_2(n, -)) \\
&\quad \text{by quasi-Yoneda lemma [29] Proposition 1 (e)} \\
&\cong \mathbf{2Alg}^i(\mathbf{T}_1, \mathbf{Cat})(F^*(\mathbf{T}^{\mathbf{op}}_2(m, -)), F^*(\mathbf{T}^{\mathbf{op}}_2(n, -))) \\
&\quad \text{since } F^* \text{ is a biequivalence, Proposition 6} \\
&\cong \mathbf{2Alg}^i(\mathbf{T}_1, \mathbf{Cat})(\mathbf{T}^{\mathbf{op}}_1(m, -), \mathbf{T}^{\mathbf{op}}_1(n, -)) \\
&\quad \text{since } F^* \text{ is an adjoint biequivalence} \\
&\simeq \mathbf{T}^{\mathbf{op}}_1(m, n) \quad \text{by quasi-Yoneda lemma [29] Proposition 1 (e)}
\end{aligned}$$

where \simeq means equivalence. \square

Many coherence results simply fall out of Proposition 7. For example, from the fact that \mathbf{T}_{Mon} is biequivalent to \mathbf{T}_{sMon} and Propositions 6 and 7 we have that every monoidal category is tensor equivalent to a strict monoidal category. More exotic statements can be asserted about higher cells in $\mathbf{2Alg}(\mathbf{T}_{Mon}, \mathbf{Cat})$.

4 Quillen Model Category Structure

In this section, we shall show that the category of $\widetilde{\mathbf{2Theories}}$ has a functorial closed Quillen model category (FCQMC) structure. We must point out that we are (perhaps wrongly) ignoring the higher categorical structure in the 3-category of $\widetilde{\mathbf{2Theories}}$. We will only talk of the (1-)category of 2-theories and 2-theory-morphisms. There will be more about this omission in Section 6.

A category is given a FCQMC structure by describing three subclasses of morphisms in the category (weak equivalences, fibrations and cofibrations) and showing that they satisfy certain axioms.

Definition 10 Weak equivalence are 2-theory-morphisms that are biequivalences. **Fibrations** are 2-theory-morphisms $F : \mathbf{T}_1 \longrightarrow \mathbf{T}_2$ satisfying the iso-2-cell lifting property. That is, for all $f \in {}_{\mathbf{1}}\mathbf{T}_1$ and for all iso-2-cells $\beta : Ff \xRightarrow{\sim} g$ in \mathbf{T}_2 there is an iso-2-cell $\alpha : f \xRightarrow{\sim} f'$ in \mathbf{T}_1 such that $F(\alpha) = \beta$. **Cofibrations** are 2-theory-morphisms that are injective on 1-cells. $F : \mathbf{T}_1 \longrightarrow \mathbf{T}_2$ are trivial fibrations (resp. trivial cofibrations) if F is both a fibration (resp. cofibration) and a weak equivalence.

Since any 2-theory-morphism $F : \mathbf{T}_1 \longrightarrow \mathbf{T}_2$ is bijective on 0-cells, one can describe these classes of maps by looking at the 1-functors $\dot{F} : \mathbf{T}_1(m, n) \longrightarrow \mathbf{T}_2(m, n)$ for all $m, n \in \mathbf{N}$. F is a weak equivalence iff the \dot{F} 's are equivalences of categories. F is a fibration iff the \dot{F} 's have the isomorphisms lifting property. F is a cofibration iff the \dot{F} 's are injective on 0-cells. Furthermore, since a product of 2-theory-morphisms is a weak equivalence (resp. fibration, cofibration) iff each of its terms is a biequivalence (resp. fibration, cofibration) and since $\mathbf{T}(m, n) = \mathbf{T}(1, n)^m$ we need only look at $\dot{F} : \mathbf{T}_1(1, n) \longrightarrow \mathbf{T}_2(1, n)$ for all $n \in \mathbf{N}$ in order to classify F . (For the reader who likes the language of operads, as opposed to 2-theories, we have just reduced the problem of classifying 2-theory-morphisms to be a problem of classifying 2-operad-morphisms. 2-operads are operads in \mathbf{Cat} . With an understanding of this paragraph, one can simply rewrite the entire paper in the language of operads). This FCQMC structure is closely associated to Rezk's [23] FCQMC structure on \mathbf{Cat} .

In order to have a FCQMC structure on the category $\widetilde{\mathbf{2Theories}}$, these three subclasses of morphisms must satisfy the following five axiom schemes.

Limits and Colimits. $\widetilde{\mathbf{2Theories}}$ must have all finite limits and colimits. These (co)limits are constructed like (co)limits in \mathbf{Cat} and $\mathbf{Theories}$. It is worth pointing out that the initial 2-theory is $\overline{\mathbf{Fin}}_{sk}$. The terminal 2-theory, \mathbf{T}_t , has $\mathbf{T}_t(m, n) = \{*\}$ for all $m, n \in \mathbf{N}$. $\mathbf{2Alg}(\mathbf{T}_t, \mathbf{Cat})$ has only one object. The algebras of $\mathbf{T}_1 \coprod \mathbf{T}_2$ are categories with both \mathbf{T}_1 and \mathbf{T}_2 structures. The algebras of $\mathbf{T}_1 \times \mathbf{T}_2$ are categories with either a \mathbf{T}_1 or a \mathbf{T}_2 structure. \square

Two out of Three. If F, G or $G \circ F$ are 2-theory morphisms and any two of them are biequivalences, then so is the third. If F and G are biequivalences then so is $G \circ F$ by Proposition 2. If $(F, F', \eta_1, \delta_1, \varepsilon_1, \zeta_1, \Phi_1, \Xi_1, \Psi_1, \Omega_1) : \mathbf{T}_1 \longrightarrow \mathbf{T}_2$

and $(G \circ F, (G \circ F)', \eta_2, \delta_2, \varepsilon_2, \zeta_2, \Phi_2, \Xi_2, \Psi_2, \Omega_2) : \mathbf{T}_1 \longrightarrow \mathbf{T}_3$ are biequivalences as in

$$\begin{array}{ccccc} & & G \circ F & & \\ & \nearrow F & & \searrow G & \\ \mathbf{T}_1 & \xrightarrow{\quad} & \mathbf{T}_2 & \xrightarrow{\quad} & \mathbf{T}_3. \\ & \nwarrow F' & & \swarrow (G \circ F)' & \end{array}$$

Then we have the following equivalence

$$(\alpha, \beta, \Gamma, \Theta) : F' \longrightarrow (G \circ F)'G$$

where

$$\begin{aligned} \alpha &= [(g \circ F)'G]\varepsilon_2 \circ_1 \eta_1 F' \\ \beta &= [(g \circ F)'G]\zeta_2 \circ_1 \delta_1 F' \\ \Gamma &= [(g \circ F)'G]\Psi_2 \circ_1 \Phi_1 F' \\ \Theta &= [(g \circ F)'G]\Omega_2 \circ_1 \Xi_1 F'. \end{aligned}$$

From this equivalence we can get the needed biequivalence $(G, G', \eta_3, \delta_3, \varepsilon_2, \zeta_2, \Phi_3, \Xi_3, \Psi_2, \Omega_2) : \mathbf{T}_2 \longrightarrow \mathbf{T}_3$ where

$$\begin{aligned} G' &= F \circ (G \circ F)' \\ \eta_3 &= F\alpha \circ \zeta_1 : id_{\mathbf{T}_2} \longrightarrow FF' \longrightarrow F \circ (G \circ F)' \circ G = G'G \\ \delta_3 &= \varepsilon_1 \circ F\beta : G'G = F \circ (G \circ F)' \circ G \longrightarrow FF' \longrightarrow id_{\mathbf{T}_2} \\ \Phi_3 &= F\Gamma \circ \Omega_1^{-1} : id \longrightarrow \varepsilon_1 \circ \zeta_1 \longrightarrow \varepsilon_1 \circ F\beta \circ F\alpha \circ \zeta_1 \longrightarrow \delta_3 \circ \eta_3 \\ \Xi_3 &= F\Theta \circ \Psi_1^{-1} : \eta_3\delta_3 = F\alpha \circ \zeta_1 \circ \varepsilon_1 \circ F\beta \longrightarrow F\alpha \circ F\beta \longrightarrow id. \quad \square \end{aligned}$$

Retracts. If F is a retract of G , that is, if there exists a commutative diagram as follows:

$$\begin{array}{ccccc} & & id & & \\ & \nearrow H & & \searrow J & \\ \mathbf{T}_1 & \xrightarrow{\quad} & \mathbf{T}_3 & \xrightarrow{\quad} & \mathbf{T}_1 \\ \downarrow F & & \downarrow G & & \downarrow F \\ \mathbf{T}_2 & \xrightarrow{\quad} & \mathbf{T}_4 & \xrightarrow{\quad} & \mathbf{T}_2 \\ & \nwarrow H' & & \swarrow J' & \\ & & id & & \end{array}$$

and if G is a weak equivalence (resp. fibration, cofibration) then F is also a weak equivalence (resp. fibration, cofibration).

Weak equivalence. If $(G, G', \eta, \delta, \varepsilon, \zeta, \Phi, \Xi, \Psi, \Omega) : \mathbf{T}_3 \longrightarrow \mathbf{T}_4$ is a weak equivalence, then so is

$$(F, JG'H, J\eta H, J\delta H, J'\varepsilon H', J'\zeta H', J\Phi H, J\Xi H, J'\Psi H', J'\Omega H') : \mathbf{T}_1 \longrightarrow \mathbf{T}_2.$$

Fibrations. Let $f \in_1 \mathbf{T}_1$ and $\beta : Ff \xrightarrow{\sim} g$ be a 2-cell in \mathbf{T}_2 . A lifting “across” F of f and β will be denoted as $L_F(f, \beta) : f \xrightarrow{\sim} f'$. The needed lifting can then

be described as

$$L_F(f, \beta) = J(L_G(H(f), H'(\beta)))$$

Cofibrations. By assumption, G, H and H' are injective on 1-cells and the left square commutes, therefore F is injective on 1-cells. \square

First Lifting Axiom. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathbf{T}_1 & \xrightarrow{U} & \mathbf{T}_3 \\ F \downarrow & \nearrow H & \downarrow G \\ \mathbf{T}_2 & \xrightarrow{V} & \mathbf{T}_4 \end{array}$$

where F is a cofibration and G is a fibration. The first axiom asserts that if F is also a weak equivalence, then there exists a lifting H making the two triangles commute.

Since F is a trivial cofibration, F is an inclusion of a full sub-2-category. By Proposition 4(1), we can construct an $F' : \mathbf{T}_2 \rightarrow \mathbf{T}_1$ such that $F' \circ F = id_{\mathbf{T}_1}$. Define H on 1-cells as follows: Let $f \in_1 \mathbf{T}_2$. Then $VFF'(f)$ is in the image of G and by the commutativity of the square is equal to $GUF'(f)$. From the biequivalence of F there is an iso-2-cell $\gamma_f : GUF'(f) \xrightarrow{\sim} V(f)$. Since G is a fibration, there is a iso-2-cell in \mathbf{T}_3 , $\delta_f : UF'(f) \xrightarrow{\sim} H(f)$. Use this as a definition of H on 1-cells.

Let $\alpha : f \rightarrow f'$ be a 2-cell in \mathbf{T}_2 . Then define H on 2-cells as

$$H(\alpha) = \delta_{f'} \circ UFF'(\alpha) \circ \delta_f^{-1} : H(f) \xrightarrow{\sim} UF'(f) \rightarrow UF'(f') \xrightarrow{\sim} H(f').$$

Such an H satisfies our requirements. \square

Second Lifting Axiom. Let F be a cofibration and G be a fibration as in the previous axiom. The second lifting axiom states that if G is also a weak equivalence, then an H exists making the triangles commute.

By assumption, F is injective on 1-cells and G is surjective on 1-cells. By a simple diagram chase, there is an $H : \mathbf{T}_2 \rightarrow \mathbf{T}_3$ defined on 0-cells and 1-cells. From the fact that G is a biequivalence and Proposition 6, $G : \mathbf{T}_3(m, n) \rightarrow \mathbf{T}_4(m, n)$ is an equivalence. For any $f, f' : m \rightarrow n \in_1 \mathbf{T}_1$, we have from the properties of an equivalence that $G : \mathbf{T}_3(H(f), H(f')) \rightarrow \mathbf{T}_4(V(f), V(f'))$ is an isomorphism. Use this isomorphism as a definition for extending the lifting H to 2-cells. \square

First Factorization Axiom. Every 2-theory-morphism $F : \mathbf{T}_1 \rightarrow \mathbf{T}_2$ can be factored as a trivial cofibration $K : \mathbf{T}_1 \rightarrow \mathbf{T}_3$ followed by a fibration $G : \mathbf{T}_3 \rightarrow \mathbf{T}_2$. \mathbf{T}_3 is the categorical version of a path space.

The 0-cells of \mathbf{T}_3 are - of course - the natural numbers. It suffices to describe the 1-cells of \mathbf{T}_3 as 0-cells of $\mathbf{T}_3(1, m)$ for all $m \in \mathbf{N}$. The 0-cells of $\mathbf{T}_3(1, m)$

are triples (f, α, g) where $f \in_0 \mathbf{T}_1(1, m)$, $g \in_0 \mathbf{T}_2(1, m)$ and $\alpha : Ff \xrightarrow{\sim} g \in_1 \mathbf{T}_2(1, m)$. 2-cells in \mathbf{T}_3 are defined as

$$\mathbf{T}_3((f, \alpha, g), (f', \alpha', g')) = \mathbf{T}_1(f, f').$$

$K : \mathbf{T}_1 \longrightarrow \mathbf{T}_3$ is defined as

$$K(f) = (f, id_{Ff}, Ff) \quad K(\gamma : f \longrightarrow f') = \gamma : (f, id_{Ff}, Ff) \longrightarrow (f', id_{Ff'}, Ff').$$

$G : \mathbf{T}_3 \longrightarrow \mathbf{T}_2$ is defined as

$$G((f, \alpha, g)) = g \quad G(\gamma : (f, \alpha, g) \longrightarrow (f', \alpha', g')) = \alpha' \circ G\gamma \circ \alpha^{-1} : g \longrightarrow g'.$$

The factorization is clear. Obviously $K : \mathbf{T}_1(1, m) \longrightarrow \mathbf{T}_3(1, m)$ is full, faithful and injective on 1-cells. For every $(f, \alpha, g) \in_1 \mathbf{T}_3(1, m)$ we have the isomorphism

$$id_f : K(f) = (f, id_{Ff}, Ff) \xrightarrow{\sim} (f, \alpha, g).$$

So K is dense and hence a trivial cofibration. As for G being a fibration, let $(f, \alpha, g) \in_2 \mathbf{T}_3$ and $\gamma : g \xrightarrow{\sim} g' \in \mathbf{T}_2$. Then $id_f : (f, \alpha, g) \xrightarrow{\sim} (f, \gamma \circ \alpha, g')$ is a lifting of γ . \square

Second Factorization Axiom. Every 2-theory-morphism $F : \mathbf{T}_1 \longrightarrow \mathbf{T}_2$ can be factored as a cofibration $K' : \mathbf{T}_1 \longrightarrow \mathbf{T}_4$ followed by a trivial fibration $G' : \mathbf{T}_4 \longrightarrow \mathbf{T}_2$. \mathbf{T}_4 is the categorical version of a mapping cylinder.

The 0-cells of \mathbf{T}_4 are again - of course - the natural numbers. It suffices to describe the 1-cells of \mathbf{T}_4 as 0-cells of $\mathbf{T}_4(1, m)$ for all $m \in \mathbf{N}$.

$$(\mathbf{T}_4(1, m))_0 = (\mathbf{T}_1(1, m))_0 \coprod (\mathbf{T}_2(1, m))_0.$$

Warning: it is not necessarily the case that $(\mathbf{T}_4(n, m))_0 = (\mathbf{T}_1(n, m))_0 \coprod (\mathbf{T}_2(n, m))_0$. The structure of $\mathbf{T}_4(n, m)$ is generally more complex but can be calculated from $\mathbf{T}_4(1, m)$. The 1-cells of $\mathbf{T}_4(1, m)$ are given as

$$T_4(f, f') = \begin{cases} \mathbf{T}_2(F(f), F(f')) & : \text{ if } f, f' \in_1 \mathbf{T}_1 \\ \mathbf{T}_2(F(f), f') & : \text{ if } f \in_1 \mathbf{T}_1, f' \in_1 \mathbf{T}_2 \\ \mathbf{T}_2(f, F(f')) & : \text{ if } f \in_1 \mathbf{T}_2, f' \in_1 \mathbf{T}_1 \\ \mathbf{T}_2(f, f') & : \text{ if } f, f' \in_1 \mathbf{T}_2 \end{cases}$$

The cofibration $K' : \mathbf{T}_1 \longrightarrow \mathbf{T}_4$ is described by

$$K'(f) = f \quad K'(\alpha : f \longrightarrow f') = F(\alpha) : F(f) \longrightarrow F(f')$$

The fibration $G' : \mathbf{T}_4 \longrightarrow \mathbf{T}_2$ is described by

$$G'(f) = \begin{cases} F(f) & : \text{ if } f \in_1 \mathbf{T}_1 \\ f & : \text{ if } f \in_1 \mathbf{T}_2 \end{cases}$$

and $G(\alpha) = \alpha$.

The factorization is clear. The fact that K' is a cofibration is obvious. G' is fibration because G' has the (unique) lifting property. G is also surjective on 1-cells and locally full and faithful. \square

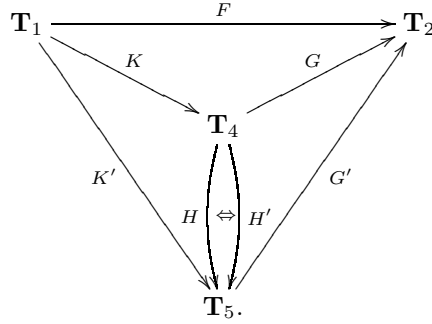
Theorem 1 *The category of 2-theories and 2-theory-morphisms admits a functorial closed Quillen model category structure.*

By inverting the weak equivalences, we get the category $Ho(\widetilde{\mathbf{2Theories}})$ and the functor $\gamma : \mathbf{2Theories} \longrightarrow Ho(\widetilde{\mathbf{2Theories}})$ which satisfies the universal property stated in the introduction..

5 Universal Properties of Coherence

The following proposition states the universal properties of the mapping cylinder formed in the Second Factorization Axiom.

Proposition 8 *Let $F : \mathbf{T}_1 \longrightarrow \mathbf{T}_2$ and let $G \circ K : \mathbf{T}_1 \longrightarrow \mathbf{T}_4 \longrightarrow \mathbf{T}_2$ be the factorization constructed in the second factorization axiom. For all factorizations $G' \circ K' : \mathbf{T}_1 \longrightarrow \mathbf{T}_5 \longrightarrow \mathbf{T}_2$ of F consisting of a cofibration followed by a trivial fibration, there is a unique **isomorphism class** of 2-theory-morphisms $H : \mathbf{T}_4 \longrightarrow \mathbf{T}_5$ making the triangles in the following diagram commute:*



That is, for every factorization $G' \circ K' = F$ there is an $H : \mathbf{T}_4 \longrightarrow \mathbf{T}_5$ such that $H \circ K = K'$ and $G' \circ H = G$. If there is any other H' that satisfies these properties, then for all $f \in_1 \mathbf{T}_4$ there is an iso-2-cell $\alpha_f : H(f) \xrightarrow{\sim} H'(f)$ such that $\alpha \circ K = K'$ and $G' \circ \alpha = G$. Note that all H and H' 's are trivial fibrations.

Remark: Proposition 8 is not a statement about all FCQMC structures since, in general, model categories do not necessarily have 2-cells.

Proof. From the Second Lifting Axiom and from the fact that the $G \circ K = F = G' \circ K'$ we have at least one H making the necessarily diagram commute.

$$\begin{array}{ccc}
 \mathbf{T}_1 & \xrightarrow{K'} & \mathbf{T}_5 \\
 K \downarrow & \nearrow H & \downarrow G' \\
 \mathbf{T}_4 & \xrightarrow{G} & \mathbf{T}_2
 \end{array}$$

From the fact that $G' \circ H = G$, both G and G' are biequivalences and the Two Out of Three Axiom, we have that H is a biequivalence. Furthermore, let G_1 be the quasi-inverse of G' . Then for all $f \in_1 \mathbf{T}_4$, $H(f)$ is isomorphic to $(G_1 \circ G)(f)$ say by

$$\beta_f : H(f) \xrightarrow{\sim} (G_1 \circ G)(f).$$

For any other $H' : \mathbf{T}_4 \longrightarrow \mathbf{T}_5$ that satisfies the commutativity of the triangles, we also have that $H'(f)$ is isomorphic to $(G_1 \circ G)(f)$ say by

$$\beta'_f : H'(f) \xrightarrow{\sim} (G_1 \circ G)(f).$$

We can now define

$$\alpha_f = (\beta'_f)^{-1} \circ \beta_f : H(f) \xrightarrow{\sim} (G_1 \circ G)(f) \xrightarrow{\sim} H'(f)$$

α_f satisfies the necessary requirements. \square

Similar universal properties can be said for the First Factorization Axiom.

In order to see Proposition 8 in action, let us work-out a concrete example. let $\mathbf{T}_1 = \mathbf{T}_{Bin}$ be the 2-theory of anomic multiplicative categories. That is, categories with a bifunctor and no associating isomorphism assumed. $\mathbf{T}_{Bin}(1, 2)$ has one element (the bifunctor) and $\mathbf{T}_{Bin}(1, 4)$ has five distinct objects (the fourth Catalan number) with no morphisms between them. Let $\mathbf{T}_2 = \mathbf{T}_{sMon}$ be the 2-theory of strict monoidal categories. $\mathbf{T}_{sMon}(1, n) = \{*\}$ for all $n \in \mathbf{N}$ (we ignore units). \mathbf{T}_{Bin} and \mathbf{T}_{sMon} respectively represent the free-est and strictest structures one can place on a category with a bifunctor. Following the construction of the Second Factorization Axiom, we get $\mathbf{T}_4 = \mathbf{T}_{Strfsh}$ the 2-theory of starfish categories.

Definition 11 A starfish category is a category \mathbf{C} with two bifunctors $\otimes, \oplus : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$ such that

$$1) A \oplus (B \oplus C) = (A \oplus B) \oplus C$$

and

$$2) \text{ there exists a iso-natural transformation } \delta_{A,B} : A \otimes B \longrightarrow A \oplus B.$$

For the association $A \otimes (B \otimes C)$ we have the following naturality diagram:

$$\begin{array}{ccc}
 A \otimes (B \otimes C) & \xrightarrow{\delta_{A, B \otimes C}} & A \oplus (B \otimes C) \\
 \downarrow id_A \otimes \delta_{B, C} & & \downarrow id_A \oplus \delta_{B, C} \\
 A \otimes (B \oplus C) & \xrightarrow{\delta_{A, B \oplus C}} & A \oplus (B \oplus C).
 \end{array}$$

From the commutativity of this square, we have a unique map of δ s from $A \otimes (B \otimes C)$ to $A \oplus (B \oplus C)$

By similar reasoning, we can extend this to all associations. For any associated word w , let w^\otimes (resp. w^\oplus) represent the functors with only \otimes (resp. \oplus) between the letters. By naturality of δ there is a unique

$$\delta_w : w^\otimes \longrightarrow w^\oplus$$

$\mathbf{T}_{Strfsh}(1, 4)$ corresponds to the following:

$$\begin{array}{ccccc}
 & A \otimes ((B \otimes C) \otimes D) & & (A \otimes (B \otimes C)) \otimes D & \\
 & \searrow \delta \sim & & \swarrow \delta \sim & \\
 A \otimes (B \otimes (C \otimes D)) & \xrightarrow[\delta]{\sim} & A \oplus B \oplus C \oplus D & \xleftarrow[\sim]{\delta} & ((A \otimes B) \otimes C) \otimes D \\
 & & \uparrow \delta \sim & & \\
 & (A \otimes B) \otimes (C \otimes D) & & &
 \end{array}$$

Hence the name “starfish”. The definition of starfish-morphisms and starfish-natural transformations are left to the reader. It is not hard to see that every starfish category is starfish-equivalent to a monoidal category and vice versa.

Let us return to Proposition 8. The 2-theory of starfish categories is the 2-theory constructed in the Second Factorization Axiom. The 2-theory of monoidal categories also satisfies the Factorization Axiom. Putting all this together, we have

$$\begin{array}{ccccc}
 \mathbf{T}_{Bin} & \xrightarrow{F} & \mathbf{T}_{sMon} & & \\
 & \searrow K & \nearrow G & & \\
 & \mathbf{T}_{Strfsh} & & & \\
 & \searrow K' & \nearrow G' & & \\
 & \mathbf{T}_{Mon} & & &
 \end{array}$$

$H \Leftrightarrow H'$

The 1-cell in $\mathbf{T}_{Strfsh}(1, 4)$ corresponding to $A \oplus B \oplus C \oplus D$ is the only “loose end”. Every other 1-cell is forced by the requirements of the commutativity of the triangles. The multiplicity of choices where one can send that 1-cell correspond to the multiplicity of the different H ’s. However all the different places where H can take the “loose ends” are (uniquely) isomorphic. Since for all $f \in_1 \mathbf{T}_{Strfsh}$, $\mathbf{T}_{Mon}(H(f), H'(f))$ is isomorphic to

$$\mathbf{T}_{sMon}(G'H(f), G'H'(f)) = \mathbf{T}_{sMon}(G(f), G(f)) = \{*\},$$

the α of Proposition 8, is in fact unique.

\mathbf{T}_{Strfsh} is the free-est structure that can be added to \mathbf{T}_{Bin} and still be Morita equivalent to \mathbf{T}_{sMon} . \mathbf{T}_{Mon} is a type of quasi-quotient of \mathbf{T}_{Strfsh} and hence also has this property. This is a universal property of \mathbf{T}_{Mon} .

There are other types of universal properties that can be said about coherence from our point of view. We can place a FCQMC structure on the category of 1-theories and 1-theory-morphisms. The FCQMC structure is trivial and hence not very interesting in itself. However, it interacts well with the FCQMC structure on **2Theories**. For **Theories**, the weak equivalences are 1-theory-equivalences which are exactly 1-theory-isomorphisms. The fibrations are all 1-theory-morphisms and the cofibrations are all 1-theory morphisms that are injective on 1-cells. In both model categories, all objects are fibrant and cofibrant.

We have the following Proposition from [22].

Proposition 9 *Let \mathbf{C} and \mathbf{C}' be model categories and let*

$$\begin{array}{ccc} & L & \\ \mathbf{C} & \xrightleftharpoons[\perp]{} & \mathbf{C}' \\ & R & \end{array}$$

be a pair of adjoint functors. Suppose L preserves cofibrations and R preserves fibrations. Then the left Kan extension $Lan_\gamma(\gamma' \circ L)$ and the right Kan extension $Ran_{\gamma'}(\gamma \circ R)$ exists and are adjoint.

$$\begin{array}{ccc} & L & \\ \mathbf{C} & \xrightleftharpoons[\perp]{} & \mathbf{C}' \\ & R & \\ \downarrow \gamma & & \downarrow \gamma' \\ Ho(\mathbf{C}) & \xrightleftharpoons[\perp]{} & Ho(\mathbf{C}') \\ & Lan_\gamma(\gamma' \circ L) & \\ & Ran_{\gamma'}(\gamma \circ R) & \end{array}$$

Placing this in our context, only the $U \vdash d$ adjunction of Diagram (1) satisfies the requirements of Proposition 9. And so we have

$$\begin{array}{ccc}
 \widetilde{\mathbf{Theories}} & \begin{array}{c} \xrightarrow{d} \\ \perp \\ \xleftarrow{U} \end{array} & \widetilde{\mathbf{2Theories}} \\
 \parallel^{id} & & \downarrow \gamma' \\
 Ho(\widetilde{\mathbf{Theories}}) & \begin{array}{c} \xrightarrow{\gamma' \circ d} \\ \perp \\ \xleftarrow{Ran_{\gamma'}(U)} \end{array} & Ho(\widetilde{\mathbf{2Theories}}).
 \end{array}$$

The other adjunctions between $\widetilde{\mathbf{Theories}}$ and $\widetilde{\mathbf{2Theories}}$ do not satisfy the requirements of the Proposition 9. Nor do any of them induce an equivalence between $\widetilde{\mathbf{Theories}}$ and $Ho(\widetilde{\mathbf{2Theories}})$. This does not prove that no such equivalence exists, but we believe the two categories are, in fact, not equivalent (How does one prove two categories are *not* equivalent?)

We would like to point to places in the literature that seem to be instances of the $\gamma' \circ d$ functor. Fangjun Arroyo [1] has proven that symmetric monoidal categories are precisely the homotopy commutative monoids in \mathbf{Cat} where the weak equivalence in \mathbf{Cat} are equivalences of categories. In our language this, in effect, becomes $\gamma' \circ d(T_{com-mon}) = [\mathbf{T}_{Sym}]$ where $T_{com-mon}$ is the 1-theory of commutative monoids and $[\mathbf{T}_{Sym}]$ is the homotopy class of the 2-theory of symmetric monoidal categories.

Recently, Tom Leinster [18] has proven a similar result for monoids and monoidal categories. We are left with the obvious question of where does braided monoidal categories fit in this scheme?

We would like to conclude by stating that if one assumes that the Kronecker bifunctor defined in Section 2 is symmetric ($\mathbf{T}_1 \otimes^K \mathbf{T}_2 \cong \mathbf{T}_2 \otimes^K \mathbf{T}_1$), then it extends to the homotopy category. All we have to do is show that \otimes^K takes two biequivalences to a biequivalence. This is a short lemma if one takes into account Proposition 3 and the way that $F_1 \otimes^K F_2$ is defined. And hence we have the following diagram which will help us build new coherence theorems from old ones.

$$\begin{array}{ccc}
 \widetilde{\mathbf{2Theories}} \times \widetilde{\mathbf{2Theories}} & \xrightarrow{\otimes^K} & \widetilde{\mathbf{2Theories}} \\
 \downarrow \gamma \times \gamma & & \downarrow \gamma \\
 Ho(\widetilde{\mathbf{2Theories}}) \times Ho(\widetilde{\mathbf{2Theories}}) & \xrightarrow{Ho(\otimes^K)} & Ho(\widetilde{\mathbf{2Theories}}).
 \end{array}$$

Example.5.1: In [29] we have shown that

$$\mathbf{T}_{sMon} \otimes^K \mathbf{T}_{sMon} \cong \mathbf{T}_{sBraid}.$$

In Section 3, we have shown that \mathbf{T}_{sMon} is Morita equivalent to \mathbf{T}_{Mon} and \mathbf{T}_{sBraid} is Morita equivalent to \mathbf{T}_{Braid} . From the above commutative square, we see that

$$\mathbf{T}_{Mon} \otimes^K \mathbf{T}_{Mon} \cong \mathbf{T}_{Braid}.$$

□

6 Future Directions

The $\widetilde{\mathbf{Theories}}$, $Ho(\mathbf{2Theories})$ Adjunction. We have shown that the usual adjunctions between $\widetilde{\mathbf{Theories}}$ and $\mathbf{2Theories}$ do not induce an equivalence between $Ho(\widetilde{\mathbf{Theories}}) = \widetilde{\mathbf{Theories}}$ and $Ho(\mathbf{2Theories})$. However, this does not mean that no such equivalence exists. Although we conjecture that the categories are in fact not equivalent, we have, as yet, no idea of how to prove this. It is left as an open question. One must realize that this question goes against the entire grain of the subject. Quillen invented model categories to show when two different model categories have the same homotopy category. We are asking to show that the different model categories have different homotopy categories.

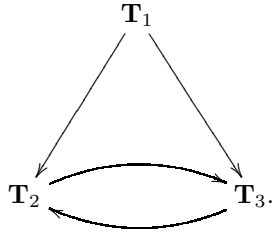
A moment of speculative thought is in order. Assume that $\widetilde{\mathbf{Theories}}$ is not equivalent to $Ho(\mathbf{2Theories})$. This would show that coherence is not simply a homotopical notion. Rather it is also in an intrinsic manner an algebraic notion. The adjunction discussed after Proposition 9 is - as all adjunctions - an algebraic concept. There are generators, relations, free and forgetful functors, universal properties etc. Coherence seems to be a complex notion which encompasses both elements of homotopy **and** elements of algebra.

The Structure – Semantics Adjunction. Between any two n -categories there are $n + 1$ different notions of equivalence that can connect them [25]. At the present time, we are looking at the diverse homotopy categories these different equivalences induce on the category of \mathbf{nCat} [30]. In particular, we shall look at the category of $\mathbf{2Cat}$. More to the point, we plan on examining the subcategory of $\mathbf{2Cat}/\mathbf{Cat}$ where the semantics of algebraic structure lives [29, 17]. Our goal will be to determine the extent to which the Structure-Semantics (quasi-)adjunction preserves the Quillen model category structures of $\mathbf{2Theories}$ and $\mathbf{2Cat}/\mathbf{Cat}$.

Higher Cells of $\mathbf{2Theories}$. By only looking at the (1-)category of 2-theories and 2-theory-morphisms, we are ignoring the higher cells of $\mathbf{2Theories}$. $\mathbf{2Theories}$ is a 3-category, but we have effectively disregarded 2-theory-natural

transformations and 2-theory-modifications (although they are used surreptitiously in the characterization of 2-theory-biequivalences.) Abandoning the higher structure seems unnatural. There are many other model categories having higher structure that should not be ignored. Surely there are important theorems that can be proved about these higher cells. To our knowledge, no one has written down axioms for a Quillen model 2-category or 3-category. How should (co)fibrations behave with respect to (iso-)2-cells? How is the fraction category constructed when there are higher cells involved? etc. The task of writing down such an axiom system is far beyond the author's capabilities. We are simply pointing to a glaring gap in the literature with the hope that someone takes on the challenge .

Relative Homotopy Theory. One of the central themes in coherence theory is that when dealing with morphisms between algebras, certain operations should be preserved up to an (iso)morphism and certain operations are preserved strictly. [29] dealt with this by having a controlling 2-theory \mathbf{T}_1 and a 2-theory-morphism $G : \mathbf{T}_1 \longrightarrow \mathbf{T}_2$ that decides what type of preservation property an operation in \mathbf{T}_2 should have. If \mathbf{T}_1 controls the operations in \mathbf{T}_2 and \mathbf{T}_3 , then a biequivalence between \mathbf{T}_2 and \mathbf{T}_3 should be strict on \mathbf{T}_1



This seems very similar to relative homotopy theory where one has a subspace (subdiagram) such that a homotopy is the identity on the subspace. Alex Heller [11] has given axioms (in the same language as [10]) for relative homotopy theory. We are planning to make the connection between relative homotopy theory and our study of coherence.

Generalizations of 2-Theories. 2-Theories can not describe all structures one usually places on a category. In particular, we must extend the definition of a 2-theory to handle contravariant functors and dinatural transformations. A method of doing this was discussed in [29]. If we are successful in formulating the notion of a generalized 2-theory, then we will be able to describe closed categories and hence enter the world of low-dimensional topology, quantum groups and computer science (e.g. traced monoidal categories). It would be nice to extend the results in this paper to generalized 2-theories. Perhaps we will be able to place a FCQMC structure on the category of 2-monads [3]. Upon entering the world of, say, low dimensional topology, we might ask what it means for one topological invariance to be of the same homotopy type as another? Similar questions in other areas are very interesting.

Bloom *et al* [4] has extended 2-theories in another direction. They have defined iteration 2-theories. These are 2-theories with extra operations that are useful in describing feedback and fixed points. See [26] for a survey of many such interesting 1-theories and 2-theories. Such generalizations is of extreme importance to computer science. They are used in describing rewrite systems, trees, data types, etc. Our goal is to extend this work to incorporate iteration 2-theories. We hope to answer questions as to when two data types are “of the same homotopy type”? When do two rewrite systems produce the same language “up to homotopy”?

Algebraic Operads. Vladimir Hinich [12] has placed a closed Quillen model category structure on the category of differential graded operads over a ring. Such operads are ways of describing algebraic structures on chain complexes of modules. One of the main ideas of quantum groups is the structure of an algebra (coalgebra, bialgebra, Hopf algebra, quasi-Hopf quasi-triangular algebra etc) A is reflected in the structure of the category of modules (comodules, bimodules, bicrossed modules etc) of A (see e.g. [16] or [31].) Hence there is some type of functor Rep from the category of differential graded operads to **2Theories** that takes an operad \mathcal{O} to the 2-theory of the structure of the category of modules from an arbitrary $A \in Alg(\mathcal{O})$. Let us explain. There are three levels of algebraic structure here. There is (i) a category of operads, $OPERADS$; (ii) for each operad $\mathcal{O} \in OPERADS$, there is a category of algebras/models of A , $Alg(\mathcal{O})$; and (iii) for each $A \in Alg(\mathcal{O})$ there is a category of modules of A , $Mod(A)$. An operad in $OPERAD$ determines the type of structure of (iii). Types of structures in (iii) are described by 2-theories. So we have a functor from $OPERADS$ to **2Theories**.

Questions: Can we formally describe this functor Rep ? Is there an inverse (quasi-adjoint, adjoint) of Rep ? Does Rep respect Hinich’s model structure? Will an inverse respect our model category structure? What is the relationship of homotopy theory to representation theory?

Other Notions of Weak Equivalence. Not every coherence relationship of interest is a biequivalence. \mathbf{T}_{Mon} and \mathbf{T}_{sMon} are biequivalent. However, the relationship between \mathbf{T}_{Braid} and \mathbf{T}_{Sym} is not so simple. We believe that they are quasi-biequivalent. A quasi-biequivalence is a weakening of the concept of a biequivalence where we substitute a 2-theory-quasi-natural transformation instead of a 2-theory-natural transformation. We would like to investigate other FCQMC structures on **2Theories** then the one given in this paper. Different sets of weak equivalences give different homotopy categories and hence different notions of coherence.

Tools of Homotopy Theory Once there is a FCQMC structure on a category \mathbf{C} one can explore and exploit the structure of \mathbf{C} with the powerful tools of homotopy theory. We might look at homotopy limits and colimits, homotopy Kan extensions, long exact sequences; homology etc. We plan on going on and looking at **2Theories** with these tools. Some further questions

arise: Are there “minimal models” of a homotopy class of 2-theories? Although there are no Postnikov towers for **2Theories** (it is not a *pointed* FCQMC), can we nevertheless decompose algebraic 2-theories “up to homotopy”. Much work remains to be done.

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